Transition Maths and Algebra with Geometry

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Lecture Notes
Electrical and Computer Engineering

Contents

Linear mappings: definition

Definition

Let V and W be two vector spaces over a field $\mathbb{K}$. A mapping $F : V \to W$ is called a *linear mapping* if it satisfies the following conditions:

- $F(v_1 + v_2) = F(v_1) + F(v_2)$ for any $v_1, v_2 \in V$,
- $\lambda \cdot F(v) = F(\lambda \cdot v)$ for any $v \in V$ and $\lambda \in \mathbb{K}$.

Let $V = W = \mathbb{R}$. Any linear mapping from $\mathbb{R}$ to $\mathbb{R}$ is of the form

$$y = a \cdot x$$

for some $a \in \mathbb{R}$.
Standard examples of linear mappings: rotation by a given angle (in more dimiensions), length multiplication etc.

Linear mappings: basic facts

Theorem

For any linear mapping $F : V \to W$ we have

$$F(0) = 0.$$

Proof...

Theorem

Any mapping $F : V \to W$ is linear iff

$F(\lambda \cdot v_1 + v_2) = \lambda \cdot F(v_1) + F(v_2)$ for any $v_1, v_2 \in V$ and any $\lambda \in \mathbb{K}$.

Proof...

Some more examples and counterexamples

Example

The following mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ is linear:

$$f(x, y, z) = (x + y, z - x)$$

Example

The following mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ is not linear:

$$f(x, y, z) = (x + y, 2z + 1)$$

Linear mappings $F : \mathbb{K}^n \to \mathbb{K}^m$

Here, we will only focus on $F : \mathbb{K}^n \to \mathbb{K}^m$. The general case is similar, yet symbolically more complicated. We will write all our vectors as columns.

Theorem

Let $A \in M_n^m(\mathbb{K})$ be an $m \times n$ matrix over $\mathbb{K}$. The mapping $F : \mathbb{K}^n \to \mathbb{K}^m$ defined for any $v \in \mathbb{K}^n$ by

$$F(v) = A \cdot v$$

is a linear mapping.

Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

Let us write an explicit formula for mapping $F(v) = A \cdot v$.

Linear mappings $F : \mathbb{K}^n \to \mathbb{K}^m$

Here, we will only focus on $F : \mathbb{K}^n \to \mathbb{K}^m$. The general case is similar, yet symbolically more complicated.

Theorem

Let $F : \mathbb{K}^n \to \mathbb{K}^m$ be a linear mapping and let $M(F)$ be a matrix defined by

$$M(F) = (F(1, 0, \dots, 0)^T, F(0, 1, \dots, 0)^T, \dots, F(0, \dots, 0, 1)^T).$$

Then

$$F(v) = M(F) \cdot v.$$

Proof - homework!

Example

Consider the linear mapping $F : \mathbb{R}^3 \to \mathbb{R}^2$ given by:

$$F(x, y, z) = (x + y, z - x)$$

Let's find $M(F)$.

Linear mappings and matrices: examples

Given a vector $\mathbf{v} \in \mathbb{K}^n$ the product $A\mathbf{v}$ can be thought of as the image of $\mathbf{v}$ under the mapping A.

Example

Fix $\alpha \in [0, 2\pi)$ and consider the matrix $R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

What is the image $R_\alpha\mathbf{v}$ for a vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$?

Example

Consider the matrix $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$. What is the image $A\mathbf{v}$ for a vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$?

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Eigenvalue: definition(s)

Definition

Let A be a $n \times n$ matrix over the field $\mathbb{K}$. A scalar $\lambda \in \mathbb{K}$ is called *eigenvalue* of A if there is a non-zero vector $\mathbf{v} \in \mathbb{K}^n$ such that

$$A\mathbf{v} = \lambda \cdot \mathbf{v}.$$

Equivalently, the definition can be restated as follows.

Definition

Let A be a $n \times n$ matrix over the field $\mathbb{K}$. A scalar $\lambda \in \mathbb{K}$ is called *eigenvalue* of A if

$$det(A - \lambda \cdot I) = 0.$$

Eigenvalue: definition(s)

Proof:
" $\Rightarrow$ ": Let $\mathbf{v}$ be a non-zero vector such that for $\lambda \in \mathbb{K}$ satisfies:

$$A\mathbf{v} = \lambda \cdot \mathbf{v}.$$

Hence,

$$A\mathbf{v} - \lambda \cdot \mathbf{v} = \mathbf{0},$$
$$(A - \lambda \cdot I) \cdot \mathbf{v} = \mathbf{0}$$

This means that the system $(A - \lambda \cdot I)X = \mathbf{0}$ has a non-zero solution. This is only when $det(A - \lambda \cdot I) = 0$.

Eigenvalue: definition(s)

Proof:
"$\Leftarrow$": Assume that for $\lambda \in \mathbb{K}$ we have $det(A - \lambda \cdot I) = 0$. This means that there is a non-zero solution to the system $(A - \lambda \cdot I)X = \mathbf{0}$. Let v be this non-zero solution. Then

$$(A - \lambda \cdot I) \cdot \mathbf{v} = \mathbf{0},$$
$$A\mathbf{v} - \lambda \cdot \mathbf{v} = \mathbf{0},$$
$$A\mathbf{v} = \lambda \cdot \mathbf{v}.$$

Eigenvectors

Definition

A vector $\mathbf{v} \in \mathbb{K}^n$ is called *an eigenvector* for an eigenvalue λ if

$$A \cdot \mathbf{v} = \lambda \cdot \mathbf{v}.$$

Theorem

Let λ be an eigenvalue of A. The set W_λ of all eigenvectors for λ is a subspace of $\mathbb{K}^n$.

Proof: Note that $\mathbf{0} \in W_\lambda$. Moreover, for $\mathbf{v_1}, \mathbf{v_2} \in W_\lambda$ we see that

$$A(\mathbf{v_1} + \mathbf{v_2}) = A\mathbf{v_1} + A\mathbf{v_2} = \lambda \cdot \mathbf{v_1} + \lambda \cdot \mathbf{v_2} = \lambda \cdot (\mathbf{v_1} + \mathbf{v_2}).$$

Hence, $\mathbf{v_1} + \mathbf{v_2} \in W_\lambda$. Similarily we prove that for $\mathbf{v} \in W_\lambda$, the vector $k \cdot \mathbf{v}$ belongs to W_λ for any $k \in \mathbb{K}$.

Example 1

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix} \quad A - \lambda \cdot I = \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{pmatrix}$$

We calculate

$$det(A - \lambda \cdot I) = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{vmatrix} =$$

$$(1-\lambda) \cdot \begin{vmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda) \cdot ((-1-\lambda) \cdot (1-\lambda) - 3) =$$

$$(1-\lambda) \cdot (\lambda^2 - 4) = (1-\lambda) \cdot (\lambda - 2) \cdot (\lambda + 2).$$

The eigenvalues of A are $1, 2, -2$.

Example 1

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{pmatrix}$$

For $\lambda = 1$ the eigenvectors are solutions to the following equation:

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & -2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \cdot X = \mathbf{0}$$

Example 1

After row reduction we get an equivalent system:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot X = \mathbf{0}$$

The solution space (the eigenvector space for $\lambda = 1$) is:

$$W_1 = \{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \}$$

Example 1

For $\lambda = 2$ the eigenvectors are solutions to the following equation:

$$\begin{pmatrix} -1 & 0 & 2 \\ 0 & -3 & 3 \\ 0 & 1 & -1 \end{pmatrix} \cdot X = \mathbf{0}$$

After row reduction we get the following equivalent system:

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \cdot X = \mathbf{0}$$

Example 1

The solution space (the eigenvector space for $\lambda = 2$) is:

$$W_2 = \{ \begin{pmatrix} 2 \cdot z \\ z \\ z \end{pmatrix} \mid z \in \mathbb{R} \}$$

Example 1

Finally $\lambda = -2$ the eigenvectors are solutions to the following equation:

$$\begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix} \cdot X = \mathbf{0}$$

After row reduction we get the following equivalent system:

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \cdot X = \mathbf{0}$$

Example 1

The solution space (the eigenvector space for $\lambda = -2$) is:

$$W_{-2} = \{ \begin{pmatrix} -\frac{2}{3} \cdot z \\ -3z \\ z \end{pmatrix} \mid z \in \mathbb{R} \}$$

Remarks

Fact

For a $n \times n$ matrix A there are at most n different eigenvalues of A.

Remark

It may happen so that a matrix A has no eigenvalues over a field $\mathbb{K}$ or some eigenvalues are multiple ones.

Example 2

Consider the matrix A over the field $\mathbb{R}$:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then:

$$det(A - \lambda \cdot I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

The equation $\lambda^2 + 1 = 0$ has no solution over $\mathbb{R}$.

Example 3

Consider the 3×3 matrix I:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then:

$$det(I - \lambda \cdot I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3.$$

The only solution is $\lambda = 1$.

Basic properties

Fact

If λ is an eigenvalue of A then it also is an eigenvalue of A^T.

Proof...

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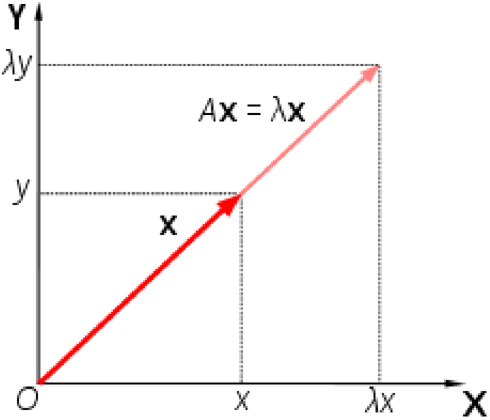
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Web graphs

Web graph:
Web pages = nodes,
Links = edges (arrows).

Web graph example and stochastic matrix

$$\begin{tikzcd} 1 \arrow[r] & 2 \\ 3 \arrow[u] \arrow[ur] & 4 \arrow[l] \arrow[u] \arrow[ul] \end{tikzcd}$$

Definition

In a web graph G for a vertex v let l_v denote the number of outgoing edges with a starting point v.

In our example $l_1 = 1, l_2 = 0, l_3 = 2, l_4 = 3$.

Definition

For any web graph G define its *stochastic matrix* S whose ij-th entry s_{ij} equals $\frac{1}{l_i}$ whenever there is a link from i to j otherwise $s_{ij} = 0$. If $l_i = 0$ then we put $s_{ii} = 1$.

Web graph example and stochastic matrix

$$
\begin{tikzcd}
1 \arrow[r] & 2 \\
3 \arrow[u] \arrow[ur] & 4 \arrow[u] \arrow[ul] \arrow[l]
\end{tikzcd}
$$

$$
S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}
$$

Stochastic matrix

Fact

For a stochastic matrix S of a web graph G we have

- $0 \leq s_{ij} \leq 1$ for any ij
- $S \cdot \begin{pmatrix} 1 \\ \ldots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \ldots \\ 1 \end{pmatrix}$
- $\lambda = 1$ is an eigenvalue of S (and S^T).

Stochastic matrix

Let w be vector whose i-th entry contains the value of probability that a surfer visits a web page i. Then w satisfies:

- $S^T w = w$,
- w has non-negative entries,
- sum of all entries in w is 1.

Problem

In general matrix S^T has many eigenvectors for $\lambda = 1$ satisfying the above properties.

Google matrix

Let α be the damping factor (e.g. $\alpha = 0.85$). Put

$$G = \alpha \cdot S + (1-\alpha) \cdot \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix} \cdot v^T$$

where v is a personalization vector with non-negative entries and sum of all entries equal to 1. It models teleportation.

Fact

The matrix G^T has a unique eigenvector for $\lambda = 1$ ($G^T \pi = \pi$) whose entries are non-negative and whose sum of all entries is 1.

Remark

Vector π in its i-th entry contains the PageRank of web page i.

All images used in this presentation come from wikipedia.org